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Monte Carlo simulations and field transformation: the scalar case[1]

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Abstract

We describe a new method in lattice field theory to compute observables at various values of the parameters λ_i in the action $S[\phi, \lambda_i]$. Firstly one performs a single simulation of a "reference action" $S[\phi^r, \lambda_i^r]$ with fixed λ_i^r. Then the ϕ^r-configurations are transformed into those of a field ϕ distributed according to $S[\phi, \lambda_i]$, apart from a "remainder action" which enters as a weight. In this way we measure the observables at values of λ_i different from λ_i^r. We study the performance of the algorithm in the case of the simplest renormalizable model, namely the ϕ^4 scalar theory on a four dimensional lattice and compare the method with the "histogram" technique of which it is a generalization.

1

1 Introduction

It is often necessary to study the observables of a given lattice field theory over a wide range of values of the parameters (coupling constants and masses). This is the case, for instance, when mapping out the phase structure of the model [1] or in problems which require a fine tuning of the parameters [2], or in the study of the lattice gauge theory beta-function [3]. In a numerical approach to these problems one should perform independent Monte Carlo simulations for each desired value of the parameters. If there are many couplings and masses this method may become prohibitively time consuming.

The so called "histogram" technique [4] is usually adopted to deal with these problems. Suppose that one wants to study observables for a lattice theory of a field ϕ with action $S[\phi, \lambda_i]$ for various values of the parameters λ_i. One runs a single Monte Carlo simulation using a reference action $S[\phi, \lambda_i^r]$ with a fixed set of values of the parameters λ_i^r. Then, by reweighting the generated configurations with the factor $e^{\Delta S}$, where $\Delta S = S[\phi, \lambda_i^r] - S[\phi, \lambda_i]$, one computes the observables associated to the action $S[\phi, \lambda_i]$.

This method is subject to various limitations [5, 6]. First of all the fluctuations of ΔS are of the order of the square root of the lattice volume. Therefore severe statistics problems are met, unless the lattice volumes are not too large. More importantly, the statistical significance and the reliability of the results depend on the size of the overlap of the regions of the configuration space covered by the importance sampling procedure based on the actions $S[\phi, \lambda_i^r]$ and $S[\phi, \lambda_i]$ respectively. As a consequence, the reliability of the results is not known a priori [5, 6].

We propose a new method to deal with these problems. Suppose that one finds a "field transformation" from ϕ, distributed according to the action $S[\phi, \lambda_i]$, to a "reference field" ϕ^r, distributed according to the "reference action" $S[\phi^r, \lambda_i^r]$. More precisely the fields ϕ and ϕ^r are related by

$$D\phi \, \exp\{-S[\phi, \lambda_i]\} \ \sim \ D\phi^r \, \exp\{-S[\phi^r, \lambda_i^r]\}, \tag{1}$$

with $D\phi$ the usual (naive) field measure. Then by a single Monte Carlo simulation with action $S[\phi^r, \lambda_i^r]$ one generates ϕ^r-configurations which can be transformed into the importance sampling configurations for ϕ.

In general, it is difficult to construct exactly the transformation (1) and one has to resort to approximations. In this case one is left with a "remainder action" $\delta S[\phi^r]$ which enters as a factor $e^{\delta S}$ in the right hand side of (1). Therefore, also in this approach, one needs to reweight the ϕ^r-configurations generated in the Monte Carlo run. The efficiency of the method depends then on: i) whether a simple enough field transformation can be constructed; ii) how large are the fluctuations introduced by the reweighting from the remainder action. iii) how large is the overlap of the ϕ– and of the transformed ϕ^r–configurations. The field transformation method is a generalization of the "histogram" technique which, in our language, corresponds to choose the trivial mapping $\phi^r = \phi$, so that the remainder action is given by $\Delta S[\phi] = S[\phi, \lambda_i^r] - S[\phi, \lambda_i]$. The advantage of our method is that one can devise a systematic procedure for improving the field transformation by reducing the "remainder action" and therefore by increasing the size of the overlap of the ϕ and of the transformed ϕ^r configurations.

We shall study the efficiency of the method in the case of the the simplest renormalizable field theory: the ϕ^4 model on a four dimensional lattice with L^4 sites. The action depends on two parameters, λ_2 and λ_4, the coefficients of the quadratic and quartic field monomials respectively. As a first example we shall consider an ultralocal field transformation, i.e. one which is the same for all lattice sites. We shall study the magnetic susceptibility χ and the second moment of the correlation function μ_2, for various values of the lattice parameters near the critical line where the square mass gap $m^2 = 8\chi/\mu_2$ vanishes. We shall consider lattice sizes $L = 8$, $L = 16$ and $L = 20$.

Particular attention will be devoted to the statistical fluctuations coming from the reweighting due to the remainder action. Since the field transformation method is a generalization of the histogram method, the analysis of the errors can be done by following the same procedure as in Ref. [5, 6].

Comparing the performance of the two methods, we find that, for the region of parameters and lattice sizes considered, the field transformation method always gives accurate values for the observables. The histogram method gives estimates of comparable accuracy only when the quartic couplings are the same, $\lambda_4^r = \lambda_4$. When this is not the case, the estimated values of the observables are not accurate although the remainder action has no large fluctuations.

In Sect. 2 we describe the method for the scalar model. It will be clear that the method can be generalized to other models. In Sect. 3 we describe the numerical performance of the simplest field transformation. In Sect. 4 we comment our results.

2 Description of the field transformation method

We consider the scalar field theory on a four dimensional lattice with periodic boundary conditions and L^4 sites. The lattice action is

$$
\begin{aligned}
S[\phi, \lambda_2, \lambda_4] &= -\sum_{n\mu} \tfrac{1}{2}\phi_n \left(\phi_{n+\mu} + \phi_{n-\mu}\right) + \sum_n v(\phi_n) \\
v(\phi_n) &= \frac{\lambda_2 + 8}{2}\phi_n^2 + \frac{\lambda_4}{4}\phi_n^4 .
\end{aligned}
\tag{2}
$$

Only two parameters λ_2 and λ_4 are needed. The normalization of the field is fixed by the kinetic term. The field ϕ_n is defined on the lattice site at the position given by the four-vector n and the sum over μ extends to four dimensions, $\mu = 1, \cdots 4$.

The Green functions, given by the expectation of field polynomials

$$
\mathrm{P}[\phi] = \phi_{n_1}\,\phi_{n_2}\cdots\phi_{n_k}\,,
$$

are defined by

$$
\langle \mathrm{P}[\phi] \rangle \equiv \frac{1}{Z}\int \prod_n d\phi_n\, e^{-S[\phi]}\,\mathrm{P}[\phi]\,, \qquad Z \equiv \int \prod_n d\phi_n\, e^{-S[\phi]}\,.
\tag{3}
$$

Our aim is to evaluate the Green functions on a range of values of the lattice parameters λ_i by using the information provided by a single Monte Carlo simulation with the reference action $S[\phi^r, \lambda_2^r, \lambda_4^r]$. To this end we look for a mapping of the field ϕ_n onto a reference field ϕ_n^r such that (1) is at least approximately satisfied. The transformation is defined by the matrix

$$
J_{nm} = \frac{\partial \phi_n}{\partial \phi_m^r}\,.
\tag{4}
$$

The field ϕ_n is a functional of the reference field, $\phi_n = \phi_n[\phi^r]$, so that also the monomial $P[\phi]$ is a functional of the reference field

$$\mathcal{P}[\phi^r] \; \equiv \; \phi_{n_1}[\phi^r] \cdots \phi_{n_k}[\phi^r] \; = \; P[\,\phi[\phi^r]\,]\,. \tag{5}$$

By changing the fields according to (4), one has

$$\prod_n d\phi_n \, e^{-S[\phi,\lambda_i]} \; = \; \prod_n d\phi_n^r \, e^{-S[\phi,\lambda_i]} \, e^{\mathrm{Tr}\ln J} \; = \; \prod_n d\phi_n^r \, e^{-S[\phi^r,\lambda_i^r]} \, e^{\delta S[\phi^r]}\,, \tag{6}$$

where the "remainder action" δS, given by

$$\delta S[\phi^r] \; \equiv \; -S[\phi,\lambda_i] \; + \; S[\phi^r,\lambda_i^r] \; + \; \mathrm{Tr}\ln J \tag{7}$$

allows for the fact that the transformation defined in (4) may not exactly satisfy (1).

The Green function (3) can be expressed in terms of weighted expectations as follows

$$\langle \mathrm{P}[\phi] \rangle \; = \; \frac{\left\langle\, \mathcal{P}[\phi^r]\, e^{\delta S[\phi^r]} \right\rangle^r}{\left\langle\, e^{\delta S[\phi^r]} \right\rangle^r}\,, \tag{8}$$

where $\langle\cdots\rangle^r$ is the expectation value computed with the action $S[\phi^r,\lambda_i^r]$.

To compute the Green functions for various values of λ_2 and λ_4, one proceeds as follows: by performing a single Monte Carlo simulation with $S[\phi^r,\lambda_i^r]$, one generates a (sufficiently large) sequence of ϕ^r-configurations. Then, for each desired value of λ_2 and λ_4, one computes, by using (4) and (7), the values of $\mathcal{P}[\phi^r]$ and $\delta S[\phi^r]$ on each configuration. Averaging over the configurations one obtains the expectation values in the numerator and denominator of (8) and therefore the Green function. In the following we discuss the form of the matrix J in (4) that we have considered.

2.1 Form of the field transformation

As stated in the introduction, an appropriate field transformation should satisfy two conflicting requirements: i) it should be easily computable; ii) it should produce a weight factor $e^{\delta S}$ with small fluctuations.

Let us consider, as a first and simplest example, the following ultralocal field transformation

$$J_{nm} = \frac{\partial \phi_n}{\partial \phi_m^r} = C \, \delta_{nm} \, e^{v(\phi_n) - v^r(\phi_n^r)} \,. \tag{9}$$

where C is a constant and v and v^r are given in (2) with parameters λ_i and λ_i^r respectively. Implementing the initial condition $\phi_n = 0$ for $\phi_n^r = 0$ the solution of (9) is

$$\int_0^{\phi_n} d\phi \, e^{-v(\phi)} = C \int_0^{\phi_n^r} d\phi \, e^{-v^r(\phi)} \,. \tag{10}$$

Since both fields are non-compact we require that $\phi_n^r \to \infty$ as $\phi_n \to \infty$. This fixes the constant C as the ratio of the two integrals extended to infinity and defines ϕ_n as a strictly monotonic function of ϕ_n^r.

From (9) we find (neglecting a field independent term)

$$\mathrm{Tr} \ln J = \sum_n [\, v(\phi_n) - v^r(\phi_n^r) \,] \,, \tag{11}$$

which gives (see (7)) as remainder action the difference of the kinetic terms

$$\delta S = \tfrac{1}{2} \sum_{n\mu} \Big(\phi_n(\, \phi_{n+\mu} + \phi_{n-\mu}) - \phi_n^r(\, \phi_{n+\mu}^r + \phi_{n-\mu}^r) \Big) \,. \tag{12}$$

This is the logarithm of the weight to be used in (8) for computing the Green functions. It is clear that too large fluctuations of the weight $e^{\delta S}$ would spoil the efficiency of the method.

In order to analyze δS we consider its expansion around small values of the fields. From (10) we obtain the following expansion of ϕ_n

$$\phi_n = C \, \phi_n^r \, \Big(1 + \mathcal{O}\big((\phi_n^r)^2 \big) \Big) \,, \tag{13}$$

which gives

$$\delta S = \tfrac{1}{2}(C^2 - 1) \sum_{n\mu} \Big(\phi_n^r(\, \phi_{n+\mu}^r + \phi_{n-\mu}^r) \Big) + \mathcal{O}\big((\phi_n^r)^4 \big) \tag{14}$$

One expects that the first term quadratic in the fields gives the largest contribution to the fluctuations. Then one may try to improve the field transformation in order to be left with a δS not containing quadratic terms in the fields.

6

As already observed, our method reduces to the usual histogram reweighting technique if one chooses the trivial mapping $\phi_n^r = \phi_n$. The corresponding remainder action becomes the difference of the potential terms $\delta S = \sum_n [\, v(\phi_n, \lambda_i) - v(\phi_n, \lambda_i^r) \,]$.

3   First application

Our aim in this section is to test how effective is the field transformation method by reproducing some standard results for the four-dimensional lattice model of the previous section.

By using the simple field transformation defined in (9) we compute for various values of λ_i the following two physical observables: the susceptibility

$$\chi(\lambda_2, \lambda_4) \;=\; \frac{1}{L^4} \sum_{n,m} \langle \phi_n \, \phi_m \rangle \;, \tag{15}$$

and the second moment of the correlation function defined by

$$\mu_2(\lambda_2, \lambda_4) \;=\; \frac{1}{L^4} \sum_{n,m} (n-m)^2 \; \langle \phi_n \, \phi_m \rangle \;. \tag{16}$$

In terms of these quantities, the mass gap squared is defined by $m^2 \equiv 8\chi/\mu_2$.

There are various points to be explored in order to test the efficiency and accuracy of the field transformation method. The remainder action is the crucial quantity to be kept under control. Indeed too large fluctuations of δS will ruin the efficiency and the accuracy of the method. The issues we consider are the following:

- the main expected challenge is the increase of the number of degrees of freedom, namely we might ask whether the quality of the results will worsen going to relatively large lattice volumes. The remainder action δS in (12) being a sum over the whole lattice, its fluctuations might grow rapidly with the square root of the volume and the method might become inefficient for large lattices;

- at a fixed lattice size and for a given reference point λ_i^r we should analyze as many values of λ_i as possible, even relatively far away from

λ_i^r. Since the (absolute value of the) remainder action δS increases with the differences $|\lambda_2 - \lambda_2^r|$ and $|\lambda_4 - \lambda_4^r|$ (see (14)), the efficiency and accuracy of the method might worsen dramatically by moving too far away from the reference parameters λ_i^r;

- finally we should test the method for λ_i close to the critical line $(\lambda_2^{\text{crit}}, \lambda_4^{\text{crit}})$ of vanishing mass gap. Here one expects that the remainder action would exhibit the large fluctuations which are typical of any quantity near a critical point.

Most computations have been performed on a 8^4 lattice, which is sufficiently small to make a fast exploration of many values of λ_i possible. A few computations on 16^4 and 20^4 lattices have been performed in order to explore the performance of the method as a function of the lattice volume.

The region of parameters explored in our analysis for the 8^4, 16^4 and 20^4 lattices is represented in Fig. 1. The reference point $r \equiv (\lambda_2^r, \lambda_4^r) = (-0.18, 0.5)$, represented as a star, indicates the values of parameters λ_2^r, λ_4^r used for the reference action. The points in the regions denoted by A, B and C correspond to the values of parameters at which we have obtained the observables by using the field transformation method. At some of these points (black circles) we also have performed independent Monte Carlo test runs. The continuous line represents the one loop estimate of the critical line $(\lambda_2^{\text{crit}}, \lambda_4^{\text{crit}})$ which, in the thermodynamic limit, is associated to the vanishing of the mass gap and to the divergence of the susceptibility. When approaching the critical points one has $(L \to \infty)$

$$\mu_2(\lambda_2, \lambda_4) \sim \chi^2(\lambda_2, \lambda_4) \to \infty \,, \qquad m^2 = 8\frac{\chi}{\mu_2} \to 0 \,, \qquad (17)$$

3.1 Field transformation

A good accuracy is necessary in the calculation of both the constant C and the functional dependence of ϕ_n on ϕ_n^r defined by (10). This is obtained by computing the two integrals $F(\phi) \equiv \int_0^\phi dt \, e^{-v(t)}$ and $F^r(\phi^r) \equiv \int_0^{\phi^r} dt \, e^{-v^r(t)}$. The constant C is obtained by $C = F(\infty)/F^r(\infty)$ and is a function of λ_i
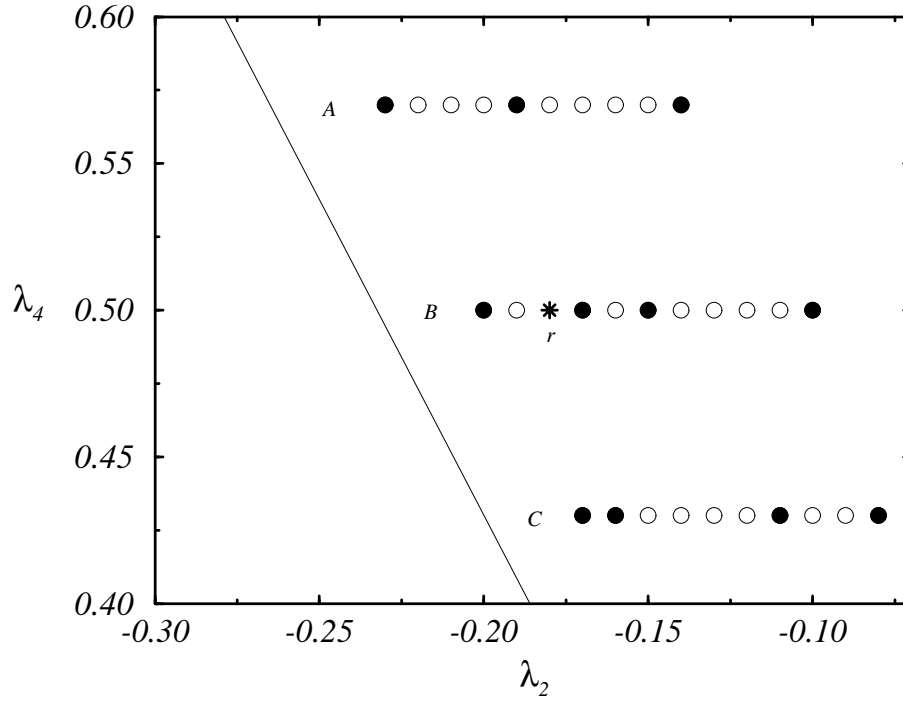
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Figure 1: *Parameter space. The continuous line indicates the one-loop critical line $\lambda_2^{\text{crit}} = -0.46479\,\lambda_4^{\text{crit}}$. The point r denotes the parameters $(\lambda_2^r, \lambda_4^r)$ of the reference action used in the Monte Carlo run. The performance of the field transformation method is studied by evaluating the observables for sets of points on the segments A, B and C (open and black circles). The black circles represent points at which Monte Carlo test runs have been performed.*

and λ_i^r. For small difference of parameters it has the form

$$C = \left(\frac{\lambda_4^r}{\lambda_4}\right)^{1/4} \left[1 - \left(\frac{(\lambda_2 + 8)^2}{\sqrt{\lambda_4}} - \frac{(\lambda_2^r + 8)^2}{\sqrt{\lambda_4^r}}\right) \frac{\pi\sqrt{2}}{\Gamma\left(1/4\right)^2} + \cdots\right] \qquad (18)$$

Then for any value of ϕ^r, the equation $F(\phi) = CF^r(\phi^r)$ is readily solved for ϕ and a table of correspondence of values for the mapping $\phi^r \rightarrow \phi$ is created. This table is then used every time a measurement is performed for the mapped configuration ϕ. The increase in CPU time is negligible compared with the direct Monte Carlo simulation.

3.2 Monte Carlo simulation

We have performed several Monte Carlo simulations with the reference action, corresponding to the point r in Fig. 1, on lattices of various sizes. We have used the Hybrid Monte Carlo updating algorithm. Each update consists in 10–15 leapfrog steps of $\Delta\tau = 0.015 - 0.05$ units of Langevin time followed by a Metropolis test with $\approx 80\%$ acceptance. We discarded approximately 10000 initial sweeps for thermalization. The measures were separated by 20–30 decorrelating sweeps. The errors were evaluated by the usual blocking procedure (see also later).

We have generated 4×10^4 configurations of the reference field ϕ^r in the thermalized regime. For a given ϕ^r-configuration, by using the field transformation (10) from λ_i^r to all considered λ_i, we have computed the value of the remainder action δS_i and of

$$\tilde{\chi}_i = \frac{1}{L^4} \sum_{nm} \phi_n\, \phi_m\,,$$

and

$$\tilde{\mu}_{2\,i} = \frac{1}{L^4} \sum_{nm} (n - m)^2\, \phi_n\, \phi_m\,.$$

Finally, by taking the appropriate average, we obtain (see (8))

$$\chi(\lambda_2, \lambda_4) \simeq \frac{\sum_i \tilde{\chi}_i \exp\{\delta S_i\}}{\sum_i \exp\{\delta S_i\}}\,, \qquad \mu_2(\lambda_2, \lambda_4) \simeq \frac{\sum_i \tilde{\mu}_{2\,i} \exp\{\delta S_i\}}{\sum_i \exp\{\delta S_i\}}\,. \qquad (19)$$

for each λ_2, λ_4. Before illustrating the results, let us discuss the fluctuations of δS.

3.3 Fluctuation of the remainder action

We plot in Fig. 2 (the wider histograms) the distribution of the remainder action δS_i for 8^4, 16^4 and 20^4 lattices and for a typical value of λ_2 while $\lambda_4 = \lambda_4^r$. The fluctuations δS_i are relatively small and therefore one expects that the statistical error of the original Monte Carlo does not substantially increase, even for the largest lattice. It is expected that the fluctuations grow as the square root of the lattice volume. Actually we find a slower growth up to $L = 20$. The width of the fluctuations for $L = 8$, $L = 16$ and $L = 20$ are respectively 0.25, 0.75, 1.05 and therefore they appear to increase almost linearly with L.

In view of possible future improvements it is interesting to examine which part of the remainder action gives the largest contribution to its fluctuations. We expand δS in powers of the field (14) and analyze the effect of the first term. In Fig. 2 we have plotted the distribution of the difference

$$\delta' S \;\equiv\; \delta S - \tfrac{1}{2}(C^2 - 1) \sum_{n\mu} \phi_n^r \, (\, \phi_{n+\mu}^r \; + \; \phi_{n-\mu}^r \,) \, . \qquad (20)$$

We see that the distributions in $\delta' S$ are distinctly narrower than those of δS and so we conclude that, in this case, most of the δS fluctuations are due to its quadratic part. The importance of the quadratic terms in δS can (partially) be understood by observing that the field typically assumes small values. For the values of λ_i considered for the distributions in Fig. 2, the values of $\phi_n^{r\,2}$ and ϕ_n^2 are typically of the order of 0.4. Therefore the terms in the expansion (14) with higher powers in the fields are expected give only a small contribution, both to δS and to its fluctuations.

For comparison, we have computed the fluctuations of the remainder action of the histogram method which is given by $\delta S = \sum_n [\, v(\phi_n, \lambda_i) - v(\phi_n, \lambda_i^r) \,]$. They are similar to those of the field transformation method as shown in Fig. 3. Here we have plotted the widths of the distributions of the remainder action δS entering in the two methods together with the width of $\delta' S$ in (20).
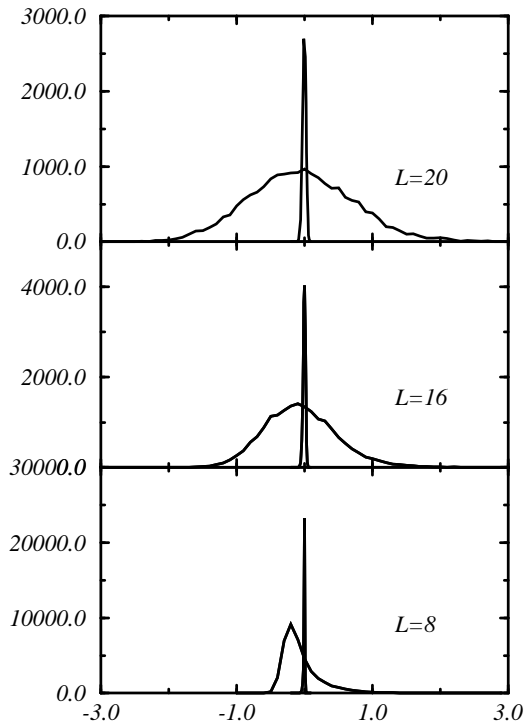
Figure 2: *Fluctuations of the remainder action for lattices of size $L = 8, 16$ and 20. The field transformation leads from the reference point r (see Fig. 1) to the point $\lambda_2 = -0.2$, $\lambda_4 = 0.5$. The wide histogram corresponds to the remainder action δS (see (12)). The narrow histogram corresponds to $\delta' S$, the remainder action minus the quadratic term (see (20)). In this example the quartic coupling is the same, $\lambda_4^r = \lambda_4$.*
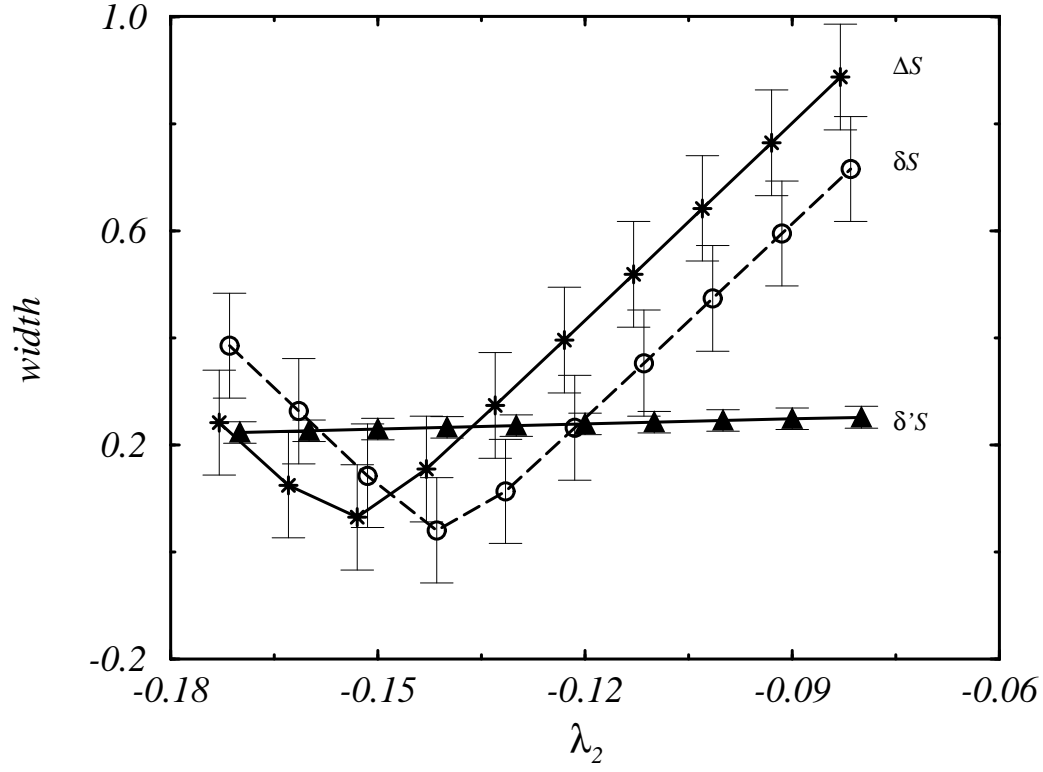
Figure 3: *The width of the fluctuations of the remainder action as a function of λ_2 for a $L=8$ lattice. The reference point is r (see Fig. 1). The quartic coupling is $\lambda_4 = 0.43$ and thus $\lambda_4^r \neq \lambda_4$. The open circles correspond to the fluctuations of δS (see (12)). The triangles stand for the fluctuations of $\delta' S$ (see (20)). The stars correspond to the fluctuations of $\Delta S = S[\phi, \lambda_2, \lambda_4] - S[\phi, \lambda_2^r, \lambda_4^r]$ (the logarithm of the weight used in the histogram method). Errors have been estimated by naïve blocking.*

13

3.4 Discussion of errors

The error analysis in the field transformation method is similar to that for the histogram method [5, 6]. Let us assume that, after *e.g.* a standard blocking, the data are uncorrelated. First of all, one must take into consideration the correlation between numerator and denominator in (19). In general, for an observable $\mathcal{F}$ given by the ratio

$$\mathcal{F} = \frac{\sum_i f_i w_i}{\sum_i w_i} \equiv \frac{\langle fw \rangle}{\langle g \rangle} \,,$$

(21)

with w_i a weight, the expression to be used for the propagation of errors is

$$\sigma_{\mathcal{F}} = \mathcal{F} \sqrt{\frac{\sigma_{fw}^2}{\langle fw \rangle^2} + \frac{\sigma_w^2}{\langle w \rangle^2} - \frac{2\sigma_{fww}^2}{\langle fw \rangle \langle w \rangle}} \,,$$

(22)

where

$$\sigma_{fw}^2 \equiv \langle f^2 w^2 \rangle - \langle fw \rangle^2$$
$$\sigma_w^2 \equiv \langle w^2 \rangle - \langle w \rangle^2$$
$$\sigma_{fww}^2 \equiv \langle fw^2 \rangle - \langle fw \rangle \langle w \rangle \,.$$

This expression explicitly shows that large fluctuations in the remainder action, giving large values of σ_w, would increase the statistical errors in the evaluation of the observables. This fact can be understood also by considering that in this case only the few configurations corresponding to the large positive fluctuations of δS would contribute to the observables.

A source of strong systematic error comes from the fact that we have sampled the Monte Carlo at the reference parameters λ_i^r with a finite set of configurations. It is known [6] that this error goes down with the logarithm of the number of configurations.

Finally we recall that in this approach, measurements at different values of the parameters are obtained from the same set of configurations and therefore all of them are strongly correlated. In practice the correlation matrix of the final measurements is almost singular. This problem is present in any kind of reweighting technique.

3.5 Results

First we consider the 8^4 lattice. In Figs. 4-6 we have plotted $1/\chi(\lambda_2, \lambda_4)$ at the values of parameters λ_i corresponding to points in the three segments A, B and C (see Fig. 1). The reported errors are obtained using (22).

In order to check the accuracy of the results, we have performed, for some values of the parameters, independent Monte Carlo simulations with the same number of configurations. The corresponding results (black circles) are reported in Figs. 4-6 together with their statistical errors. The agreement between the results obtained by the field transformation and by the independent Monte Carlo runs is good even for values of the parameters not very close to the reference point r. We have also computed by the same technique the values of $\mu_2(\lambda_2, \lambda_4)$ and found similar good agreement.

In the various plots with fixed λ_4 (on the segments A, B and C) the values of $1/\chi$ decrease with λ_2, but they tend to flatten upward before the critical value is reached, as expected since we work with a finite lattice. We also find, as expected, that μ_2 is more sensitive to the finite volume effects (see (16)).

The statistical errors obtained by the field transformation method are not really larger than those obtained by the direct Monte Carlo. This is due to the fact that, as shown in the previous subsection (see Figs. 2–3), the δS-fluctuations are not large.

Results for larger volumes. Since the remainder action is a sum over the full lattice one may expect that the efficiency of the method will worsen by increasing the lattice volume. However we have found (see Figs. 2–3) that, even for large lattices, the δS-fluctuations are not too large. Therefore the agreement with the results from a direct Monte Carlo test run should not be sizably degraded.

This expectation is confirmed by our study for larger lattices. In Figs. 7 and 8 we have plotted the values of $1/\chi(\lambda_2, \lambda_4)$ obtained for 16^4 and 20^4 lattices respectively. Like in Figs. 4-6, we have reported also the measures obtained by Monte Carlo test simulations with the same number of independent configurations. The agreement with the direct Monte Carlo results is still good. As expected, the statistical errors are larger, since the fluctuations
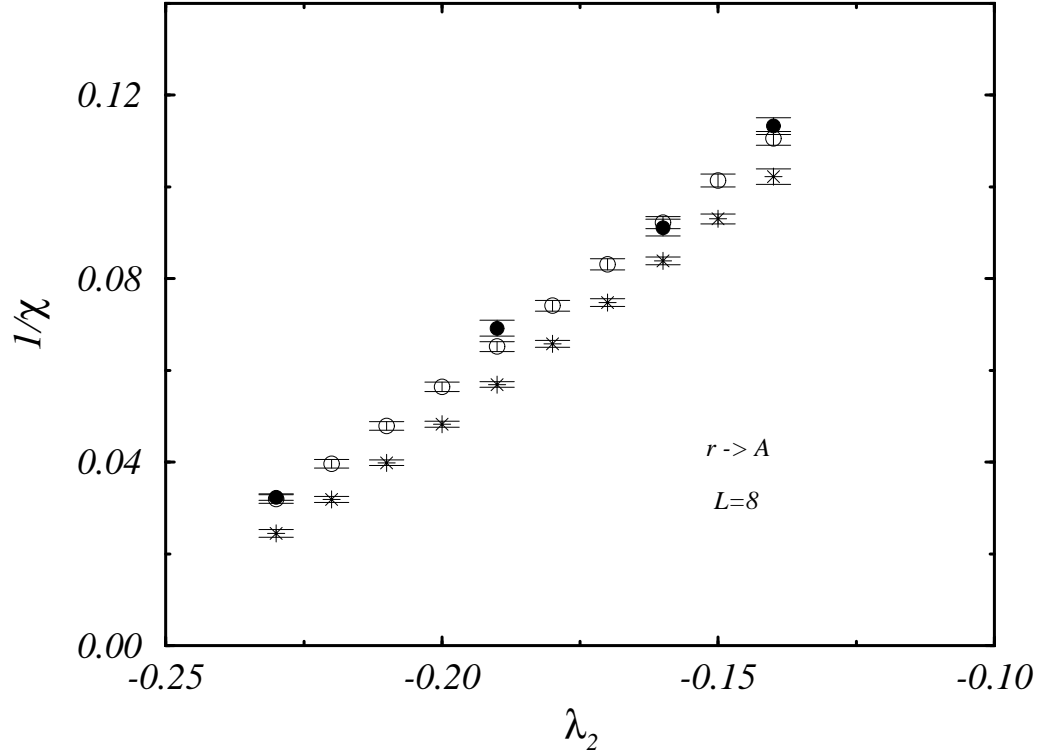
Figure 4: *Lattice of size $L = 8$. The emtpy circles represent the values of $1/\chi(\lambda_2, \lambda_4)$ obtained by the field transformation method from the reference point r to values of (λ_2, λ_4) in the segment A of Fig. 1. The errors are purely statistical and they are computed according to (22). We give also the values obtained by Monte Carlo test runs with the same statistics (black circles) and the values coming from the usual histogram method (stars).*
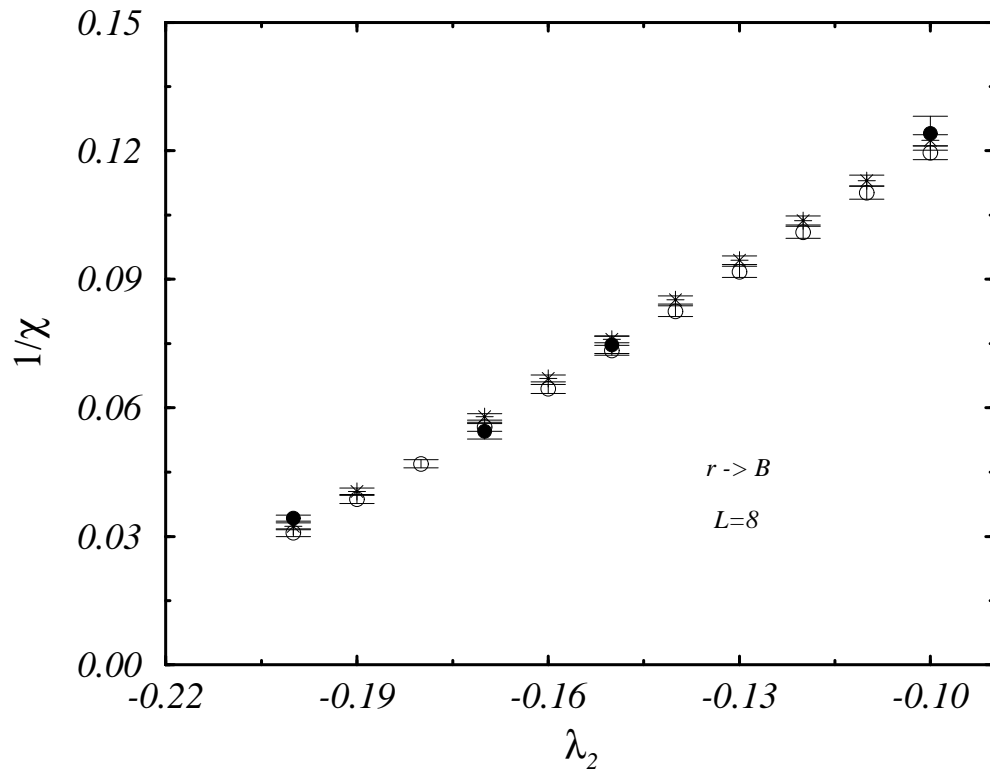
Figure 5: *Same as in Fig. 4, but for a mapping from the reference point r to values of (λ_2, λ_4) in the segment B of Fig. 1.*
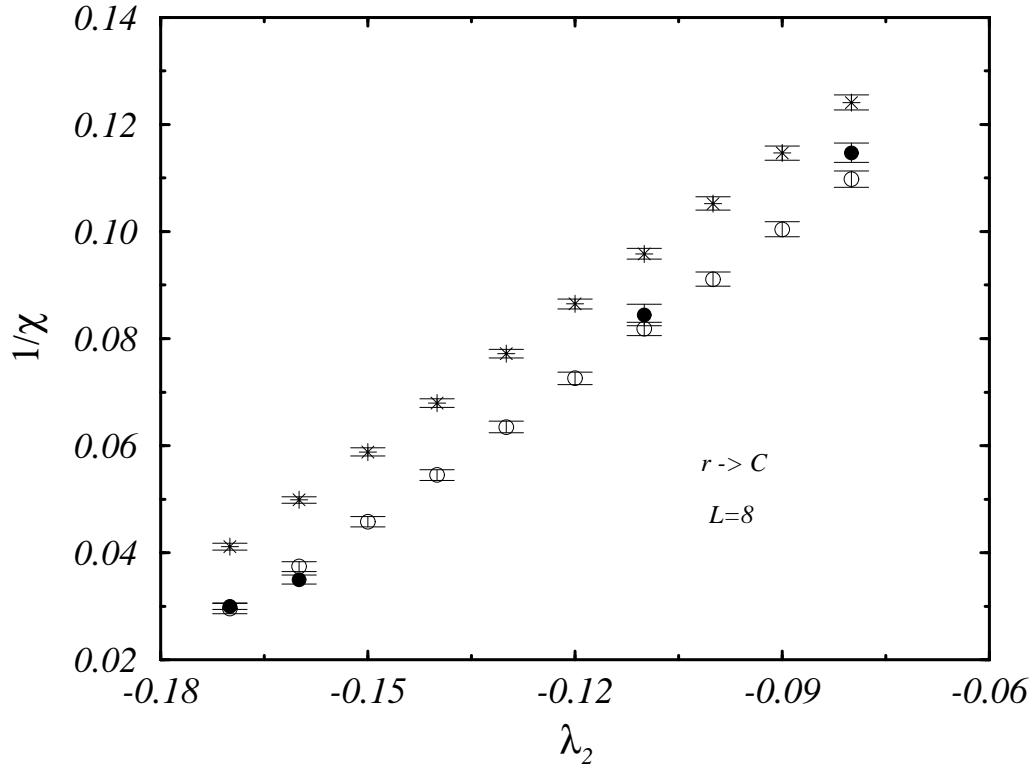
Figure 6: *Same as in Fig. 4, but for a mapping from the reference point r to values of (λ_2, λ_4) in the segment C of Fig. 1.*
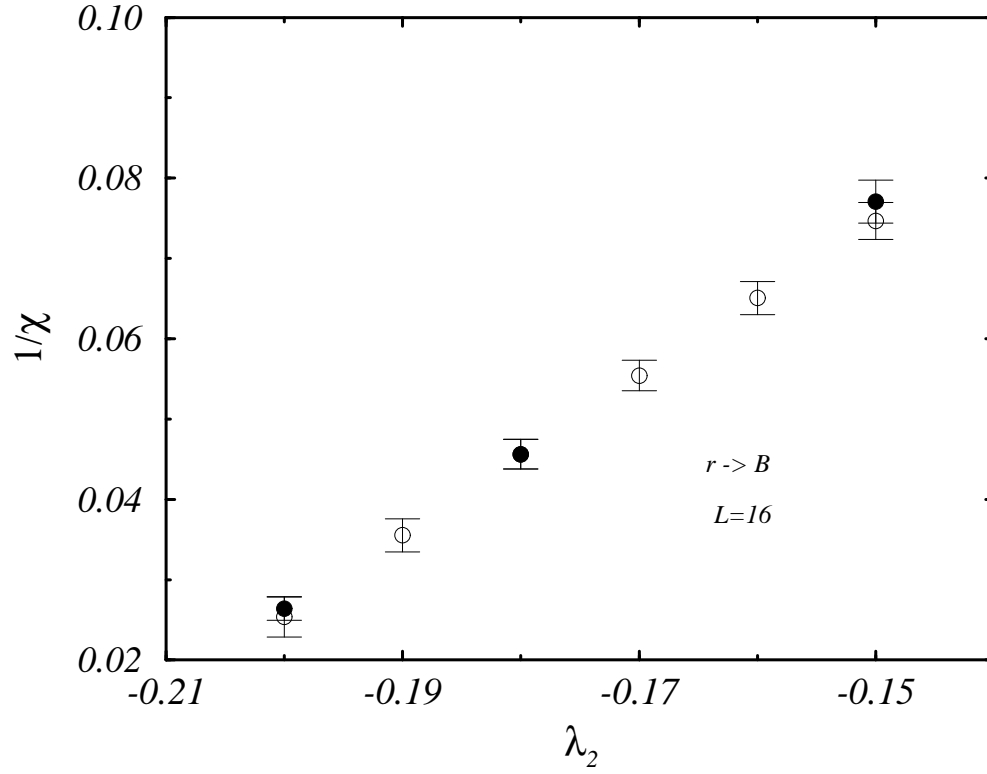
18

Figure 7: *Lattice of size L=16. We have plotted $1/\chi(\lambda_2, \lambda_4)$ obtained by the field transformation method from the reference point r to parameters (λ_2, λ_4) in the segments B (see in Fig. 1). The errors are purely statistical and they are computed according to (22). We give also the values obtained by Monte Carlo test runs with the same statistics (black circles).*

19

of the remainder action increase. However the increase of the errors is not dramatic and one still has a good determination of the observables.

Comparison with the histogram method. In Figs. 4-6 we plot also the results obtained by the histogram method (stars). We have to distinguish the case in which the quartic coupling λ_4 is the same as that of the reference action λ_4^r or is not. In the first case (see Fig. 5 in which $\lambda_4 = \lambda_4^r = 0.5$) we find that the results obtained by the Monte Carlo test simulations agree with those obtained both by the field transformation and by the histogram method.

If instead $\lambda_4 \neq \lambda_4^r$ (see Figs. 4 and 6) we find that the results of the Monte Carlo test simulations disagree with the ones of the histogram method, while they still agree with the ones of the field transformation. This indicates that for $\lambda_4 \neq \lambda_4^r$ the regions of field configuration space probed by the histogram method for the two actions $S[\phi, \lambda_i]$ and $S[\phi, \lambda_i^r]$ have a small overlap. Therefore in order to obtain a reliable result by the histogram method one has to considerably increase the number of configurations sampled. The alternative we have proposed is to improve the overlapping by a field transformation.

4   Discussion and conclusions

The field transformation method, which we have introduced and analysed, can be viewed as a generalization of the usual histogram method. In principle it has the advantage that, by a suitable choice of the mapping, one has the possibility of reducing the remainder action, the crucial quantity entering into the weight which can endanger the statistical significance of the calculation. We have analyzed the case of a simple mapping, the ultralocal one in (9). It gives a remainder action which has small fluctuations in a wide range of parameters and lattice sizes. Moreover the results for the susceptibility (and other computed quantities) obtained by the field transformation agree quite well with those obtained by Monte Carlo test runs, even when the lattice volume increases (at least up to $L = 20$, see Figs. 4-8). This implies that the region of ϕ^r-configurations sampled by the Boltzmann weight $e^{-S[\phi^r, \lambda_i^r]}$, when transformed by (9), overlaps sufficiently well with the region probed by $e^{-S[\phi, \lambda_i]}$.
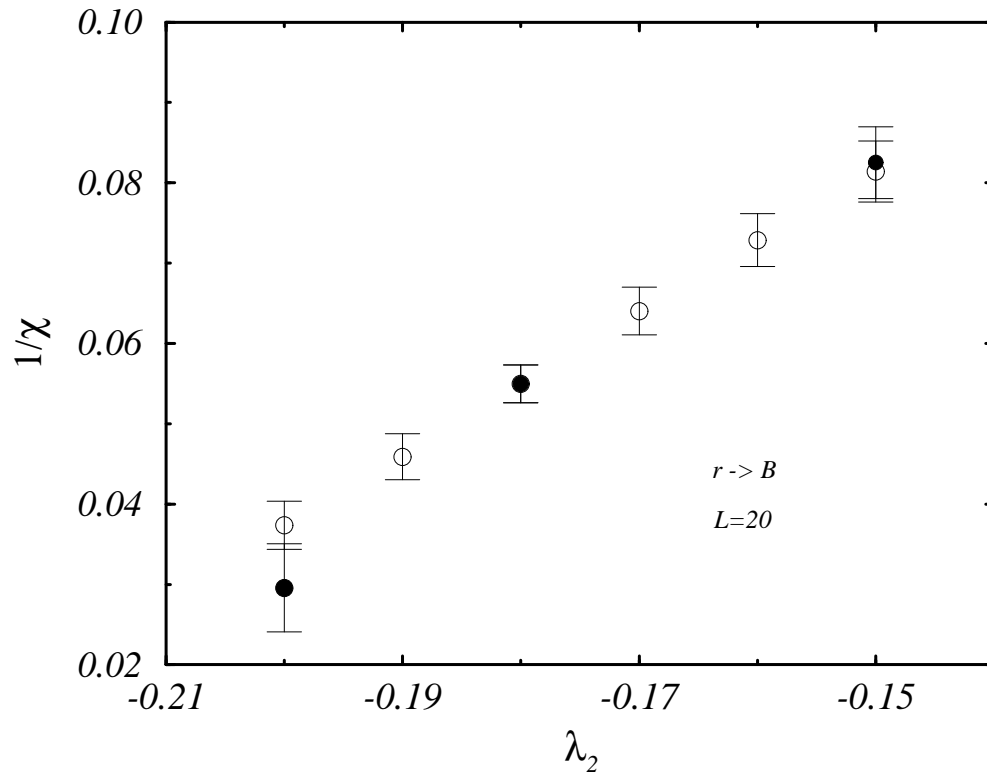
20

Figure 8: *Same as in Fig. 7 for a lattice of size L=20.*

21

Since the ultralocal transformation already gives small fluctuations and consistent results, we have not been forced to improve it. However we have indicated a possible strategy to further reduce the remainder action. The error analysis and the statistical independence of the results obtained by the field transformation method can be discussed analogously to the case of the histogram method [5, 6].

The histogram method corresponds to the trivial field transformation: $\phi_n^r = \phi_n$. For $\lambda_4 \neq \lambda_4^r$, we find that the results of this method disagree with those obtained by Monte Carlo test runs. This indicates that, for $\lambda_4 \neq \lambda_4^r$, the relevant ϕ-configuration region for the action $S[\phi, \lambda_i]$ is not sufficiently probed by this method.

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